



Fracture Mechanics

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J Integral

Path independent contour integrals

In a pioneering work, Eshelby (1956-1974) defined a number of contour integrals that were path independent by virtue of the theorem of energy conservation. This was achieved while he was studying dislocations in elastic domains.

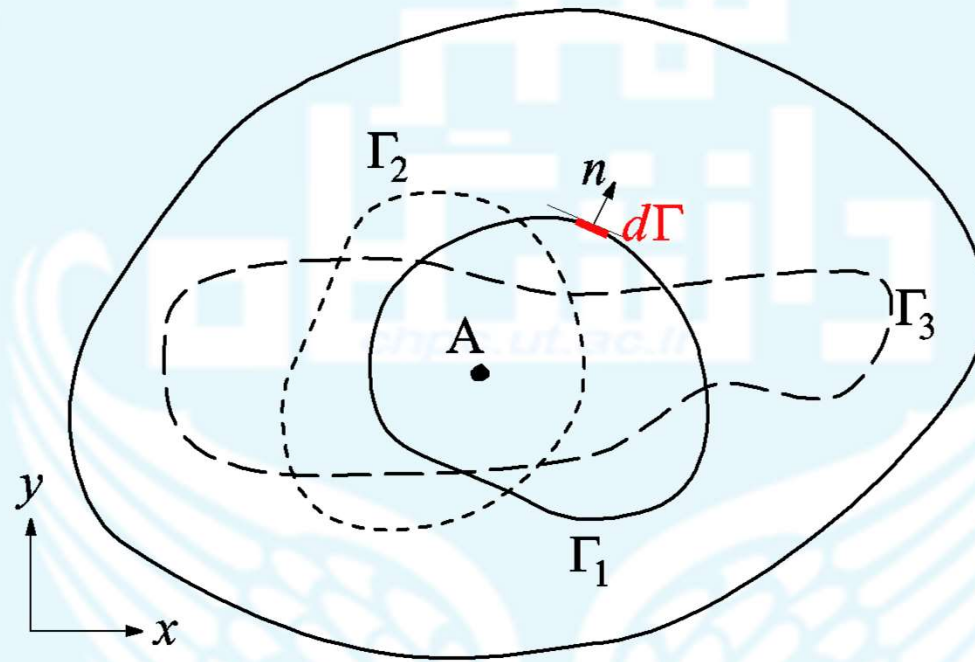
Eshelby did not realize its importance or applications in fracture mechanics. It was up to Rice and Rosengren (1968) to notice the importance of the J integral as a criterion for crack growth in fracture mechanics (Anderson, 1995).

While the original introduction of the J integral was limited to problems with no unloading and no internal stress/strains, the new developments have now been well extended to even cohesive crack and dynamic problems.

Path independency of J also allows for evaluation of linear and nonlinear elastic energy release rates and elastoplastic work far from the crack tip.

Development of the J integral allowed for computation of fracture mechanics problems in general problems. It somehow helped to save the fracture mechanics from being an obsolete academic science with limited practical mandate.

Eshelby (Eshelby 1974) defined a number of contour integrals that are path independent by virtue of the theorem of energy conservation:



$$J = \oint_{\Gamma_1} (\dots) d\Gamma = \oint_{\Gamma_2} (\dots) d\Gamma = \oint_{\Gamma_3} (\dots) d\Gamma = \text{constant}$$

First, the problem is considered without the presence of the body force. The two-dimensional form of one of these integrals can be written as

$$J = \oint_{\Gamma} \left(w_s dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$

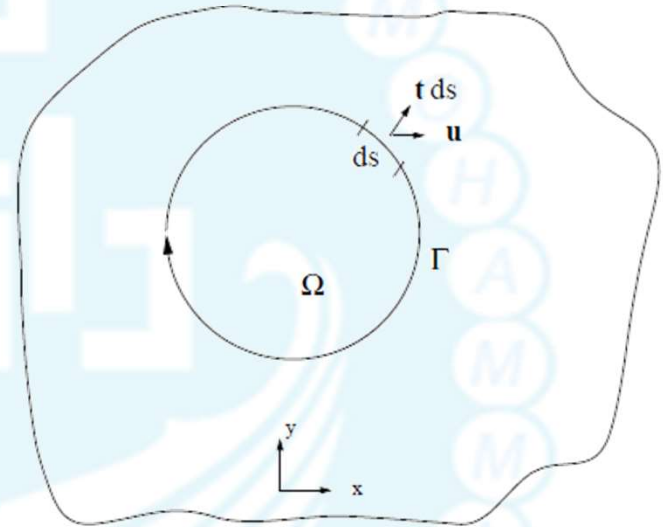
or in component form,

$$J = \oint_{\Gamma} \left(w_s \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial x} \right) n_j d\Gamma$$

where

$$w_s = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \text{ is the strain energy density}$$

$$\mathbf{t} = \boldsymbol{\sigma} \mathbf{n} \text{ is the traction vector}$$



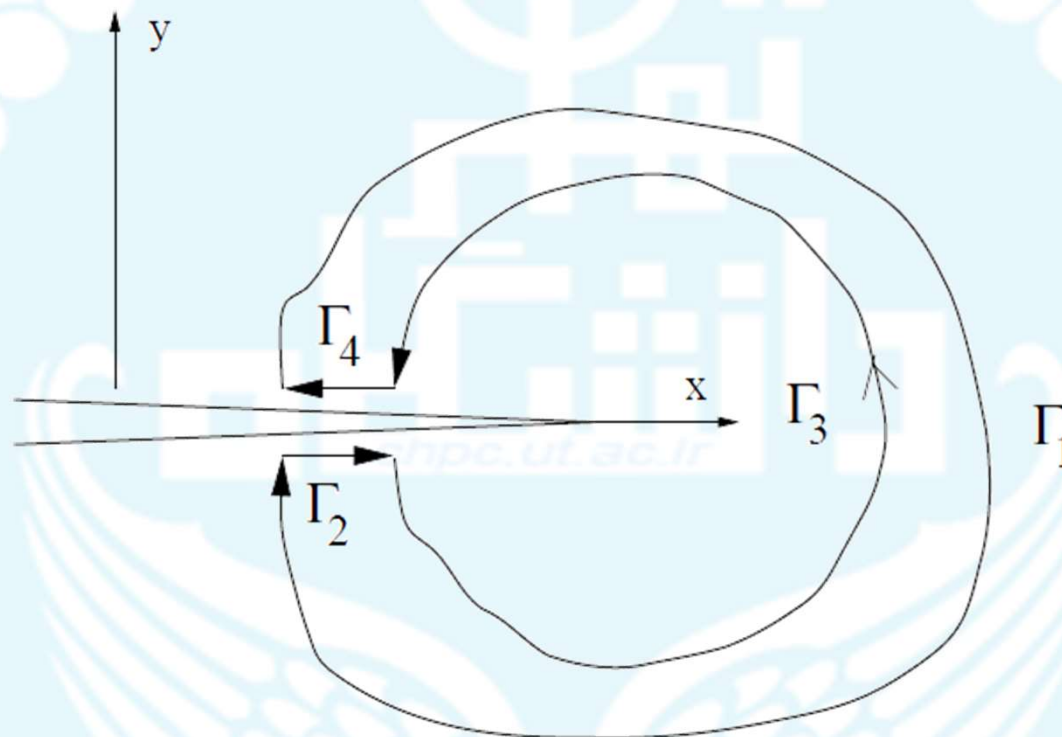
It can be proved that the Eshelby's expression for J is indeed equal to zero for a closed path:

$$J = \oint_{\Gamma} \left(w_s dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right) \quad \begin{cases} dx = -n_y d\Gamma \\ dy = -n_x d\Gamma \end{cases}$$
$$J = \oint_{\Gamma} \left(w_s n_x - n_j \sigma_{ij} \frac{\partial u_i}{\partial x} \right) d\Gamma$$

Invoking the Green's theorem, applying the chain rule, and based on the equilibrium equation in the absence of body forces, leads to:

$$J = 0$$

J integral for traction-free crack problems



$$J = \oint_{\Gamma} \left(w_s dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right), \quad \Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$$

$$J = J_{\Gamma_1} + J_{\Gamma_2} + J_{\Gamma_3} + J_{\Gamma_4} = 0$$

$$J_{\Gamma_n} = \oint_{\Gamma_n} \left(w_s dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$

For Γ_2 and Γ_4 contours:

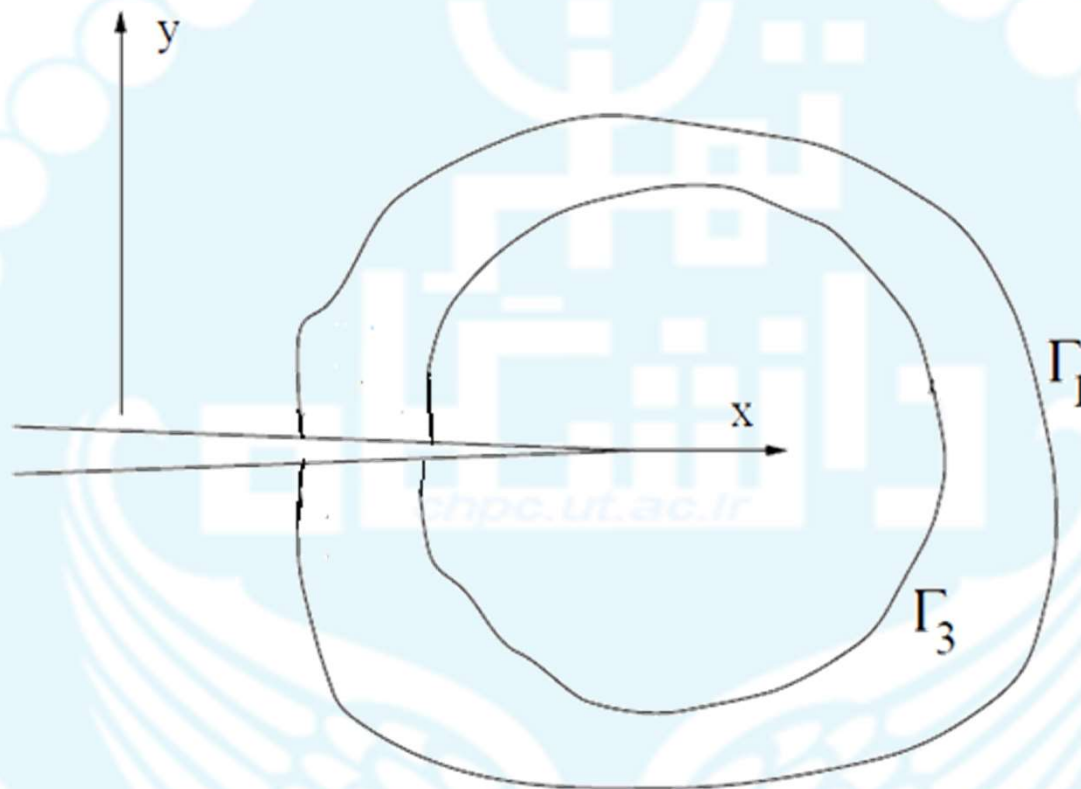
$$\mathbf{t} = 0, \quad dy = 0 \rightarrow J_{\Gamma_2} = J_{\Gamma_4} = 0$$

and

$$J = J_{\Gamma_1} + J_{\Gamma_3} = 0$$

$$J_{\Gamma_1} = -J_{\Gamma_3} \quad (= \text{constant})$$

J integral for traction-free crack problems



$$J = \int_{\Gamma} \left(w_s dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$

Therefore, it can be concluded that the absolute values of the J integral evaluated over arbitrary paths Γ_1 and Γ_3 remain identical; an indication of the path independency of J .

As a result, if a contour begins from one crack surface and ends at the other face, it can be used to determine the J integral. The constant value (of the J integral) is no longer zero.

Selection of the size and shape of the appropriate contour curve for a specific problem requires complementary numerical studies. Practically, they should also be related to the geometry of the crack and the finite element model, as will be further discussed in the following sections.

It can be shown that when J is applied along a contour around a crack tip, it represents the change in potential energy for a virtual crack extension da .

The total potential energy of a two-dimensional domain including a traction free crack that is surrounded by a contour curve Γ under quasi-static conditions and in the absence of body forces can be defined as:

$$\Pi = \int_{\Omega} W_s d\Omega - \int_{\Gamma} t_i u_i d\Gamma$$

For a virtual crack extension da , the change in potential energy is

$$\frac{d\Pi}{da} = \int_{\Omega} \frac{dW_s}{da} d\Omega - \int_{\Gamma} \left[t_i \frac{du_i}{da} + \frac{dt_i}{da} u_i \right] d\Gamma$$

Applying the divergence theorem and omitting zero terms, results in

$$-\frac{d\Pi}{da} = \int_{\Gamma} \left[W_s dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right]$$

which is similar to the definition of J-integral:

$$J = -\frac{d\Pi}{da}$$

This is equivalent to the definition of the fracture energy release for linear elastic materials, G ,

$$J = G$$

This important conclusion can be used to define the energy release rate for a unit crack extension in nonlinear elastic materials.

The conclusion $J=G$:

1. Allows a general numerical approach for evaluation of the energy release rate for general problems.
2. The computation is limited to only a finite contour, and does not require computation on the whole problem, as usually required by G .
3. The formulation is an energy-based non-local approach. As a result, it can be readily extended to multi-physics problem.
4. It can be extended to more complex problems, including body force, dynamic, traction on crack and even to anisotropic and non-homogeneous problem.
5. It is valid for nonlinear elastic materials.



Generalization of J -Integral

Generalisation of J

The original definition of J can be regarded as the first component of a more general path independent vector with the following components:

$$J_1 = \int_{\Gamma} \left\{ w_s dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right\}$$

$$J_2 = \int_{\Gamma} \left\{ w_s dx - \mathbf{t} \frac{\partial \mathbf{u}}{\partial y} d\Gamma \right\}$$

The generalised J also satisfies the following relation for crack extensions parallel and perpendicular to the crack (Hellen and Blackburn, 1975)

$$J = J_1 - iJ_2 = \frac{1}{E'} (K_I^2 + K_{II}^2 + 2iK_I K_{II})$$

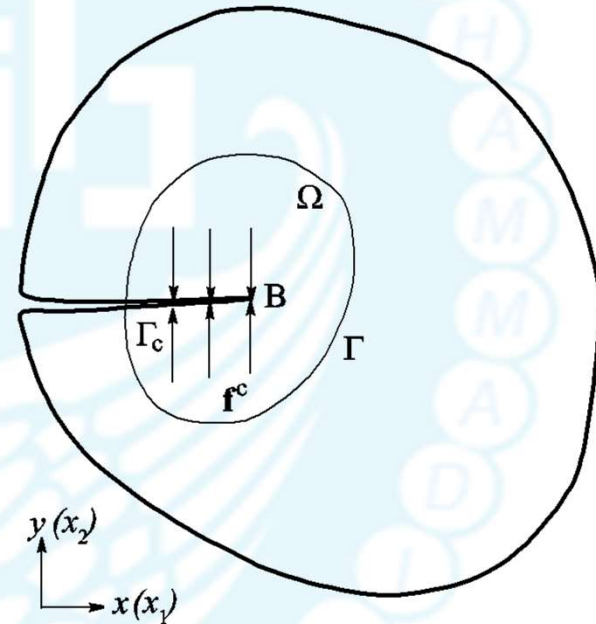
Note that J_1 and J_2 **do not represent** mode I and mode II, respectively.

Effect of crack surface traction

Karlsson and Backlund (1978) extended the concept of the J integral to account for the effect of crack surface tractions by simply extending the definition of the contour path to include the crack surfaces:

$$J = \int_{\Gamma} \left\{ w_s dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right\} - \int_{\Gamma_c} \mathbf{f}^c \frac{\partial \mathbf{u}}{\partial x} d\Gamma$$

$$J = \int_{\Gamma} \left\{ w_s n_i - t'_j u_{j,i} \right\} d\Gamma - \int_{\Gamma_c} \left\{ f_j^c u_{j,i} \right\} d\Gamma$$



Effect of body force

In the case of present body force, $\mathbf{f}^b$, the equilibrium equation can be written as

$$\sigma_{ij,j} + \mathbf{f}_i^b = 0$$

and a modified J integral has to be defined (Atluri, 1982):

$$J = \int_{\Gamma} \left\{ w_s dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right\} - \int_{\Omega} \mathbf{f}^b \frac{\partial \mathbf{u}}{\partial x} d\Omega$$

$$J = \int_{\Gamma} \{ w_s n_i - t_j u_{j,i} \} d\Gamma + \int_{\Omega} \{ -f_j^b u_{j,i} \} d\Omega$$

Effect of thermal initial strains

In thermo-elasticity problems, the strain energy density w_s must be redefined in terms of net strains and net effective stresses:

$$\bar{\sigma}'_{ij} = C_{ijkl} \bar{\varepsilon}_{kl} = C_{ijkl} (\bar{\varepsilon}_{kl} - \alpha T \delta_{kl})$$

$$W_s = \int_0^{\bar{\varepsilon}_{ij}} \bar{\sigma}'_{ij} d\bar{\varepsilon}_{ij} \xrightarrow{\bar{\sigma}'_{ij} = C_{ijkl} \bar{\varepsilon}_{kl}} W_s = \frac{1}{2} C_{ijkl} \bar{\varepsilon}_{ij} \bar{\varepsilon}_{kl}$$

and the final form of J integral becomes: $\bar{\sigma}'_{jk,k} + \mathbf{f}'^b_j = 0$

$$J = \int_{\Gamma} \left\{ w_s dy - t' \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right\} + \int_{\Omega} \left\{ \alpha \bar{\sigma}'_{jj} T_{,i} - \mathbf{f}'^b \frac{\partial \mathbf{u}}{\partial x} \right\} d\Omega$$

$$J = \int_{\Gamma} \left\{ w_s n_i - t'_j u_{j,i} \right\} d\Gamma + \int_{\Omega} \left\{ \alpha \bar{\sigma}'_{jj} T_{,i} - f'^b_j u_{j,i} \right\} d\Omega$$

Effect of pore pressure initial stresses

The strain energy density w_s must be redefined:

$$\sigma_{ij} = \sigma'_{ij} + \sigma_{ij}^0 = C_{ijkl} \varepsilon_{kl} - p \delta_{ij}$$

$$W_s = \int_0^{\varepsilon_{ij}} \sigma'_{ij} d\varepsilon_{ij} \xrightarrow{\sigma'_{ij} = C_{ijkl} \varepsilon_{kl}} W_s = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl}$$

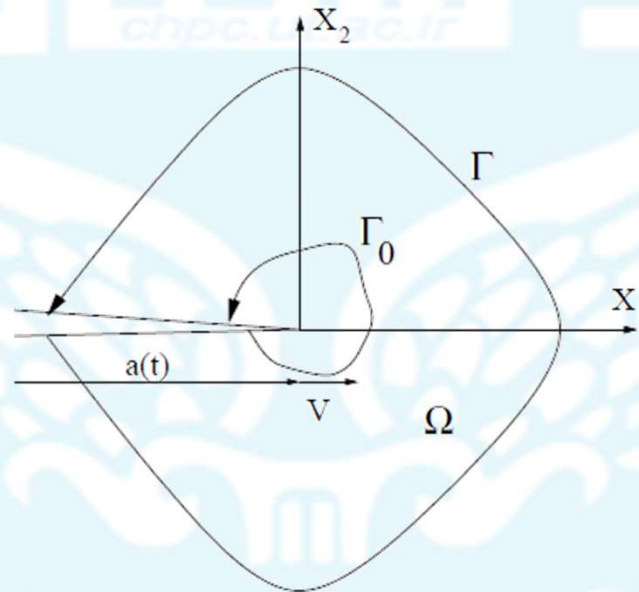
and the final form of J integral becomes: $\bar{\sigma}'_{jk,k} + \mathbf{f}'^b_j = 0$

$$J = \int_{\Gamma} \left\{ w_s dy - t' \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right\} + \int_{\Omega} \left\{ (\mathbf{p}_{,j} - \mathbf{f}'^b) \frac{\partial \mathbf{u}}{\partial x} \right\} d\Omega$$

$$J = \int_{\Gamma} \{ w_s n_i - t'_{,j} u_{j,i} \} d\Gamma + \int_{\Omega} \{ (p_{,j} - f'^b_j) u_{j,i} \} d\Omega$$

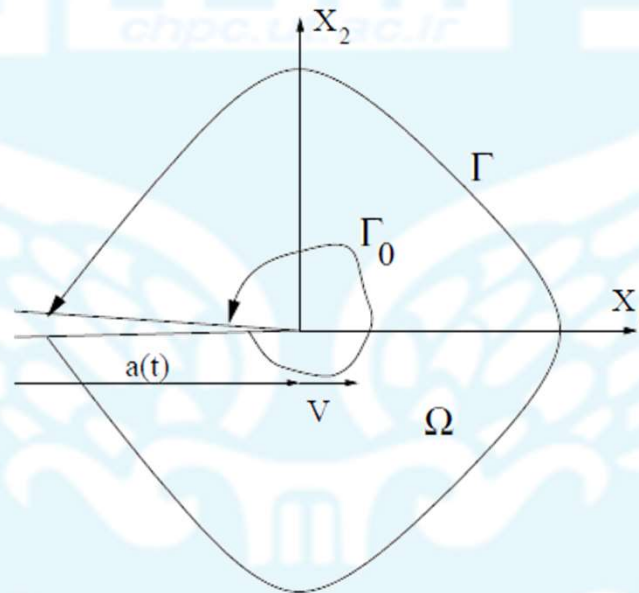
J integral in dynamic problems

Consider a body containing a propagating crack. We seek to determine the energy released to the crack tip and thus consider a vanishingly small loop Γ_0 surrounding the crack tip. Whereas the loop can have any shape, it must remain fixed relative to a coordinate system attached to the crack tip.



In contrast to the static case, the energy release rate can not be expressed by a path-independent contour integral because if a wave front intercepts one contour, but not the second one, then two different values would be represented by only the contour integral.

A is the area bounded by the outer boundary Γ , the traction free crack faces, and the inner loop Γ_0 .



The rate of work P done by the tractions on Γ is equal to the rate of increase of the strain energy U and kinetic energy T in region A and the flux F of energy into the crack tip region:

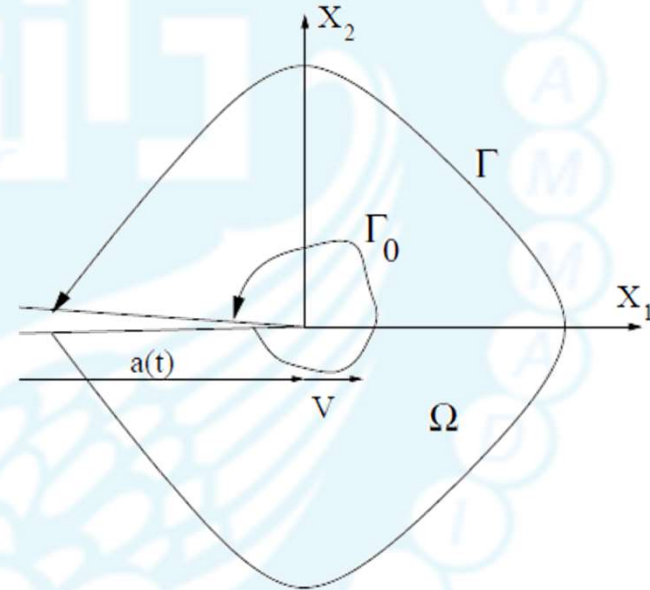
$$\dot{P} = \dot{U} + \dot{T} + \dot{F}$$

$$\dot{W} = \dot{U}_K + \dot{U}_s + \dot{U}_\Gamma$$

$$P = \int_{\Gamma} t_i u_i d\Gamma$$

$$U = \lim_{\Gamma_0 \rightarrow 0} \int_A W(\varepsilon_{ij}) dA$$

$$T = \lim_{\Gamma_0 \rightarrow 0} \int_A \frac{1}{2} \rho \dot{u}_i \dot{u}_i dA$$



Dynamic Crack Propagation

Since the loop Γ_0 moves with the crack tip, region A is time dependent. Therefore, the time rate of change of the strain energy equals the integral over A of the time rate of change of the strain energy density $W(\varepsilon_{ij})$ less the flow of this density through the loop Γ_0 .

$$\dot{U} = \lim_{\Gamma_0 \rightarrow 0} \int_A \dot{W}(\varepsilon_{ij}) dA - \lim_{\Gamma_0 \rightarrow 0} \int_{\Gamma_0} W(\varepsilon_{ij}) V_n d\Gamma$$

V_n is the component of the crack-tip velocity normal to Γ_0

$$V_n = V n_1 = V \cos \theta$$

θ is the angle that the outward unit normal to Γ_0 with x_1 .

Thus,

$$\dot{W}(\varepsilon_{ij}) = \sigma_{ij} \dot{\varepsilon}_{ij} = \sigma_{ij} \dot{u}_{i,j}$$

$$\dot{U} = \lim_{\Gamma_0 \rightarrow 0} \int_A \sigma_{ij} \dot{u}_{i,j} dA - \lim_{\Gamma_0 \rightarrow 0} \int_{\Gamma_0} W(\varepsilon_{ij}) V_n d\Gamma$$

$$\dot{T} = \lim_{\Gamma_0 \rightarrow 0} \int_A \rho \ddot{u}_i \dot{u}_i dA - \lim_{\Gamma_0 \rightarrow 0} \int_{\Gamma_0} \frac{1}{2} \rho \dot{u}_i \dot{u}_i V_n d\Gamma$$

Therefore from $\dot{P} = \dot{U} + \dot{T} + \dot{F}$,

$$\begin{aligned} F &= \lim_{\Gamma_0 \rightarrow 0} \int_{\Gamma_0} \left(W + \frac{1}{2} \rho \dot{u}_i \dot{u}_i \right) V_n d\Gamma + \int_{\Gamma} t_i \dot{u}_i d\Gamma \\ &\quad - \lim_{\Gamma_0 \rightarrow 0} \int_A \left(\sigma_{ij} \dot{u}_{i,j} + \rho \ddot{u}_i \dot{u}_i \right) dA \end{aligned}$$

Then from the equation of motion,

$$\sigma_{ij,j} = \rho \ddot{u}_i$$

the divergence theorem, and

$$\sigma_{ij} \dot{u}_{i,j} = \left(\sigma_{ij} \dot{u}_i \right)_{,j} - \sigma_{ij,j} \dot{u}_i$$

the rate at which energy is lost from the body due to flux through Γ_0 is obtained:

$$F = \lim_{\Gamma_0 \rightarrow 0} \int_{\Gamma_0} \left[\left(W + \frac{1}{2} \rho \dot{u}_i \dot{u}_i \right) V_n + t_i \dot{u}_i \right] d\Gamma$$

$$F = \lim_{\Gamma_0 \rightarrow 0} \int_{\Gamma_0} \left[(w_s + w_k) V \delta_{1j} + \sigma_{ji} \dot{u}_i \right] n_j d\Gamma$$

In an increment of time dt , the crack extends by $da = Vdt$ and the energy expended is Fdt .

Thus the energy release rate is

$$J = \frac{F}{V}$$

If we have a moving coordinate system x – y attached to the crack tip where $x=X-a(t)$ and $y=Y$, then

$$\dot{u}_i = -V \frac{\partial u_i}{\partial x} + \frac{\partial u_i}{\partial t}$$

Under steady condition, the second term is equal to zero, and close to the crack tip the displacement changes rapidly with position (at a fixed time) and the first term dominates in all cases,

$$J = \lim_{\Gamma_0 \rightarrow 0} \int_{\Gamma_0} \left[(w_s + w_k) \delta_{1j} - \sigma_{ji} \frac{\partial u_i}{\partial x} \right] n_j d\Gamma$$

$$J = \lim_{\Gamma_0 \rightarrow 0} \int_{\Gamma_0} \left[(w_s + w_k) dy - \sigma_{ji} n_j \frac{\partial u_i}{\partial x} d\Gamma \right]$$

For dynamic problems:

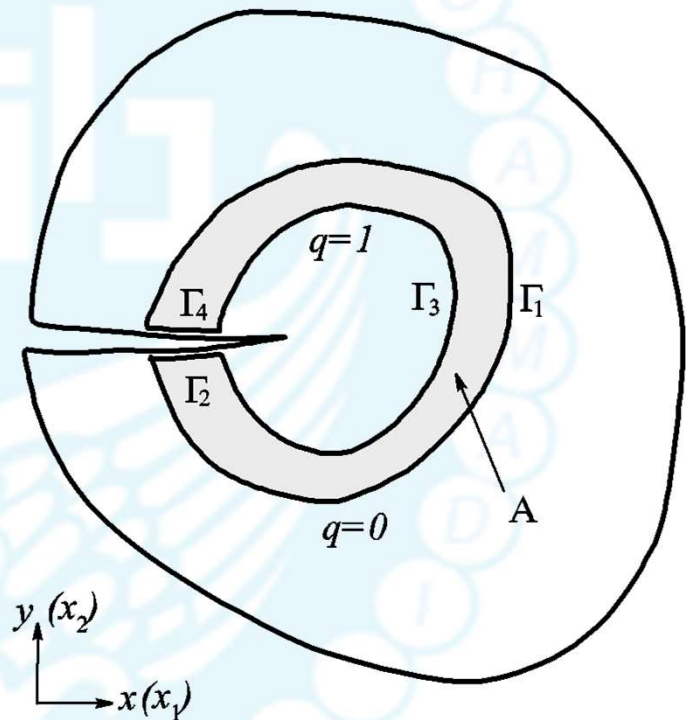
- J is contour dependent and does not remain constant for arbitrary contours.
- The contour must be selected as small as possible to reduce the effect of moving waves on evaluation of J .
- For small contours, the accuracy of stress field is low in conventional finite elements. Therefore, accurate advanced finite element formulations are required for sensitive dynamic problems.

Equivalent domain integral (EDI) method

Li *et al.* (1985) proposed the equivalent domain integral method as an alternative approach to original contour J integral.

$$J = \int_A \left[\sigma_{ij} \frac{\partial u_i}{\partial x_1} - w_s \delta_{1j} \right] \frac{\partial q}{\partial x_j} dA$$

where q is an arbitrary smoothing function which is equal to unity on Γ_3 and zero on the boundary of the J domain Γ_1 .



The function q may be interpreted as the kinematically admissible virtual extension field due to a unit extension of the crack (Li *et al.*, 1985, Moran and Shih, 1987 and Gravouil *et al.*, 2008).

In fact, the contour integral has been replaced by an equivalent area integral, which is more suited to finite element solutions.

In FEM, the inner contour Γ_3 is often taken as the crack tip, and so A naturally corresponds to the area inside Γ_1 . The boundary Γ_1 should also coincide with element boundaries to facilitate numerical calculations.

Alternatively, a fine evaluation is obtained with a q field whose spatial derivatives are zero close to the crack tip, avoiding poorly evaluated stress calculations in the vicinity of crack tip (Gravouil *et al.*, 2008).

The value of q within an element can be interpolated as:

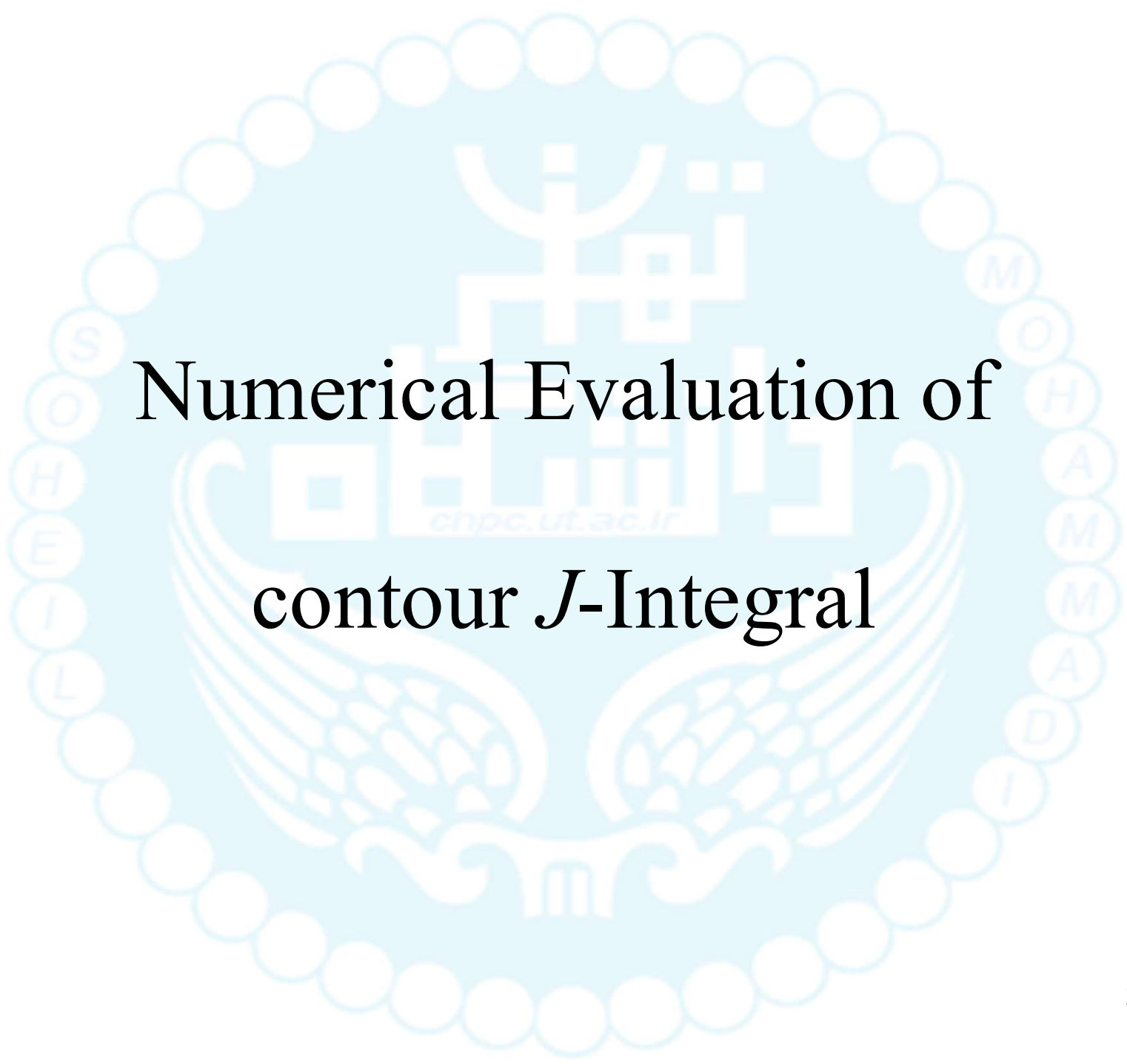
$$q = \sum_{i=1}^n N_i(x) q_i$$

where n is the number of nodes per element, q_i are the nodal values of q , and N_i are the element shape functions. Evaluation of J then follows:

$$J = \sum_{A^*} \sum_{g=1}^{ng} \left\{ \left[\left(\sigma_{ij} \frac{\partial u_i}{\partial x_1} - w_s \delta_{1i} \right) \frac{\partial q}{\partial x_i} \right] \det \left(\frac{\partial x_j}{\partial \xi_k} \right) \right\}_g W_g$$

The equivalent domain integral:

- It is very convenient and flexible for finite element solutions.
- Using different contours allows for computations to be performed on points far from the crack tip, where the accuracy of stress may be poor in conventional finite elements.
- The inner contour can be reduced to a point on crack tip. In this case, the accuracy may be reduced due to less accurate stresses near the crack tip.
- Various domains may generate different values of J due to numerical approximations.
- The method can be extended to 2.5D/3D simulations.



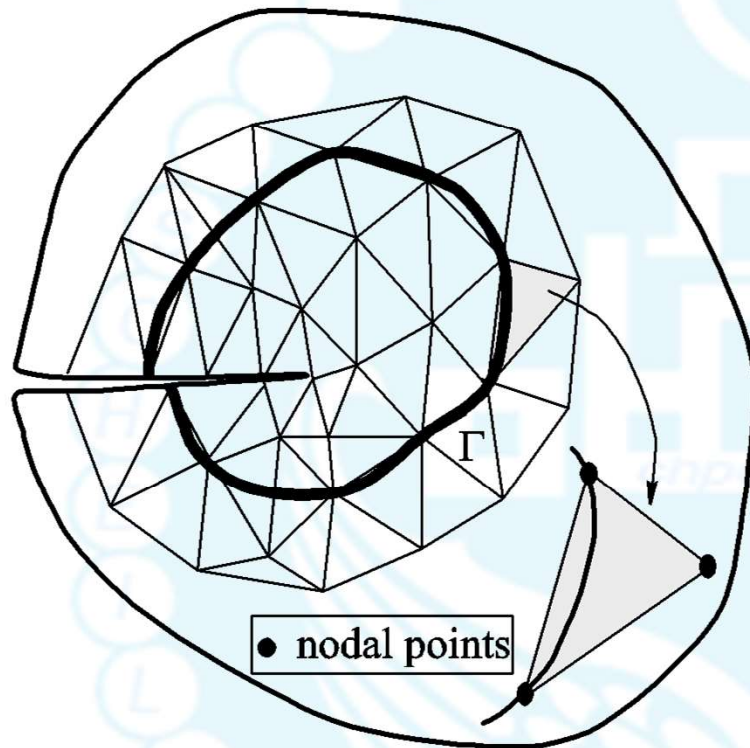
Numerical Evaluation of contour J -Integral

Numerical evaluation of the J integral

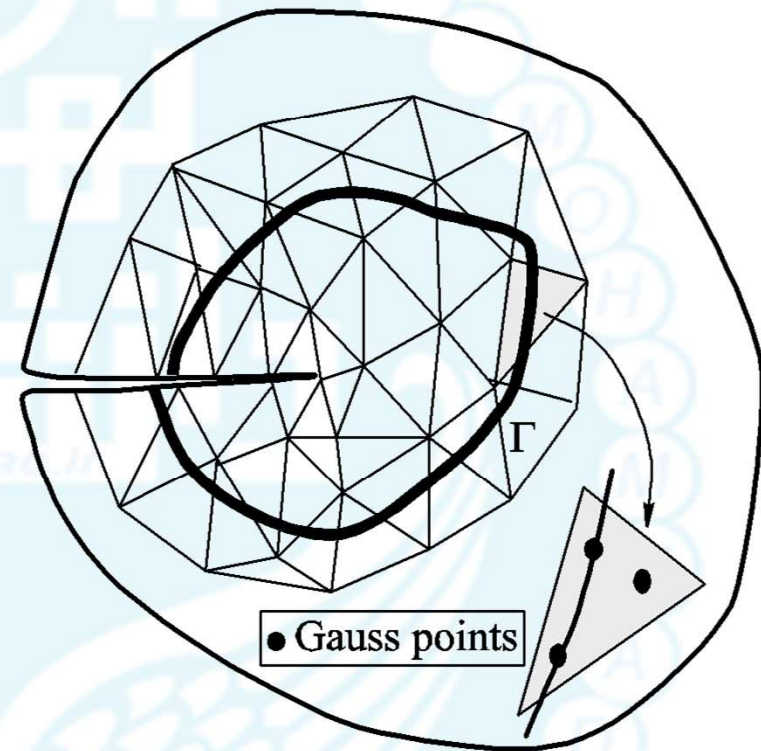
The equivalency of the J integral and G within the context of LEFM allows for evaluation of J according to the available approaches previously discussed for the energy release rate.

In this section, however, algorithms for direct evaluation of contour J are presented.

At present, most simulations are based on the direct evaluation of the J integral, which is more compatible with the structure of the finite element method. The results will then be used to determine the stress intensity factor and the energy release rate from the classical concepts of fracture mechanics.



a) contour passes through nodes



b) contour passes through Gauss points

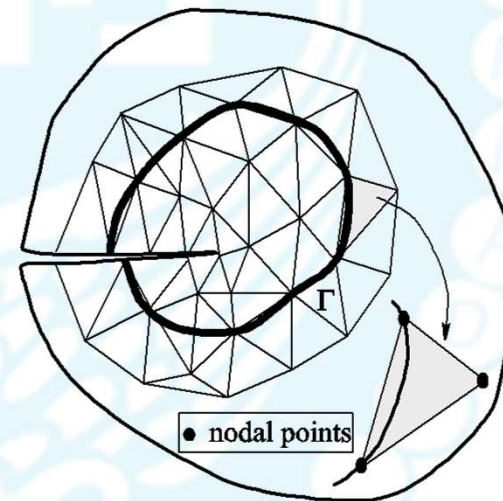
a) Nodal solution for contour J-integral

First consider the rather simple case where stresses are known at the nodes, then the numerical evaluation of J would become relatively simple.

Beginning with a modification of J for a crack along the ξ in a plane stress problem:

$$J = \int_{\Gamma} W_s d\eta - \int_{\Gamma} \mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial \xi} d\Gamma$$

$$J = \int_{\Gamma} W_s d\eta - \int_{\Gamma} (\sigma_n, \sigma_t) \begin{bmatrix} \frac{\partial \mathbf{u}_n}{\partial \xi} \\ \frac{\partial \mathbf{u}_t}{\partial \xi} \end{bmatrix} d\Gamma$$



contour passes through nodes

where

$$W_s = \frac{1}{2E} (\sigma_{xx} + \sigma_{yy})^2 + \frac{1+2\nu}{E} (\sigma_{xy}^2 - \sigma_{xx}\sigma_{yy})$$

$$\sigma_n = \sigma_{xx} \cos^2 \theta + \sigma_{xy} \sin \theta \cos \theta + \sigma_{yy} \sin^2 \theta$$

$$\tau_t = (\sigma_{yy} - \sigma_{xx}) \sin \theta \cos \theta + \sigma_{xy} (\cos^2 \theta - \sin^2 \theta)$$

$$u_n = u_x \cos \theta + u_y \sin \theta$$

$$u_t = -u_x \sin \theta + u_y \cos \theta$$

where θ is the angle between the normal to the contour and x direction at each node.

The J integral can be calculated by evaluation of the values of the following expression between adjacent nodes and computing the total sum around the contour,

$$\begin{aligned}
 J = & \int w(\sin \theta dx + \cos \theta dy) \\
 & - \int \sigma_{xx} \varepsilon_{xx} \cos \theta dy + \int \sigma_{xy} \varepsilon_{yy} \sin \theta dy - \int \sigma_{xy} \varepsilon_{xx} \cos \theta dx \\
 & - \int \sigma_{yy} \varepsilon_{yy} \sin \theta dx + \int \sigma_{xy} \varepsilon_{xy} \cos \theta dy - \int \sigma_{xy} \varepsilon_{xy} \sin \theta dx \\
 & + \int (\sigma_{xx} \sin \theta - \sigma_{xy} \cos \theta) du_n - \int (\sigma_{yy} \sin \theta + \sigma_{xy} \cos \theta) du_t
 \end{aligned}$$

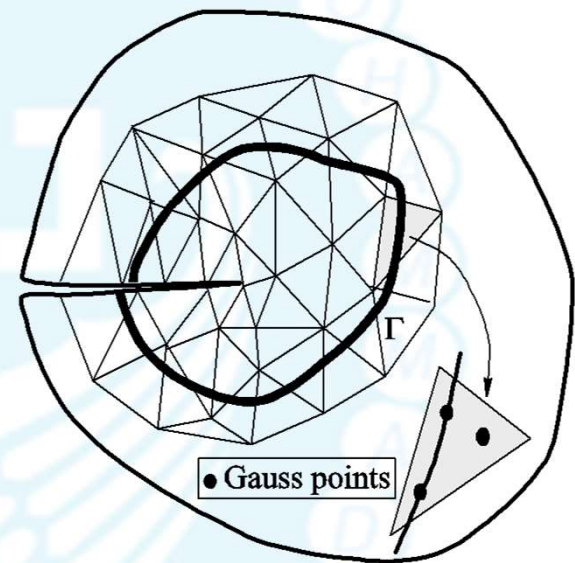
b) General finite element solution for contour J-integral

Most standard finite element codes, however, only provide Gauss point stresses, and hence care must be exercised in properly determining the J integral along a path passing through them.

Consider a crack to be along the x axis.

From the definition of J ,

$$J = -\frac{\partial \Pi}{\partial a} = \int_{\Gamma} \left(W_s dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$



contour passes through Gauss points

The path is intentionally designed so the same integration points associated with the element stiffness matrix can be used. Therefore, the contour path must pass through the finite element Gauss points. ³⁹

Defining the traction and displacement vectors, the strain energy density and the arc length in terms of the problem specifications,

$$W_s = \frac{1}{2} \left[\sigma_{xx} \frac{\partial u_x}{\partial x} + \sigma_{xy} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \frac{\partial u_x}{\partial x} + \sigma_{yy} \frac{\partial u_y}{\partial y} \right]$$

$$dy = \frac{\partial y}{\partial \eta} d\eta$$

$$\mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial x} = \left[\left(\sigma_{xx} n_1 + \sigma_{xy} n_2 \right) \frac{\partial u_x}{\partial x} + \left(\sigma_{xy} n_1 + \sigma_{yy} n_2 \right) \frac{\partial u_y}{\partial x} \right]$$

$$d\Gamma = \sqrt{\left(\frac{\partial x}{\partial \eta} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2} d\eta \quad \text{at } d\xi = \text{constant}$$

results in,

$$J = \int_{\Gamma} \left\{ \frac{1}{2} \left[\sigma_{xx} \frac{\partial u_x}{\partial x} + \sigma_{xy} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \frac{\partial u_x}{\partial x} + \sigma_{yy} \frac{\partial u_y}{\partial y} \right] \frac{\partial y}{\partial \eta} \right. \\ \left. - \left[\left(\sigma_{xx} n_1 + \sigma_{xy} n_2 \right) \frac{\partial u_x}{\partial x} + \left(\sigma_{xy} n_1 + \sigma_{yy} n_2 \right) \frac{\partial u_y}{\partial x} \right] \sqrt{\left(\frac{\partial x}{\partial \eta} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2} \right\} d\eta$$

which can be evaluated by a Gauss integration rule along the contour path Γ :

$$J = \sum_{g=1}^{n_g} W_g I_g(\xi_g, \eta_g)$$

W_g is the Gauss weighting factor

n_g is the number of integration points

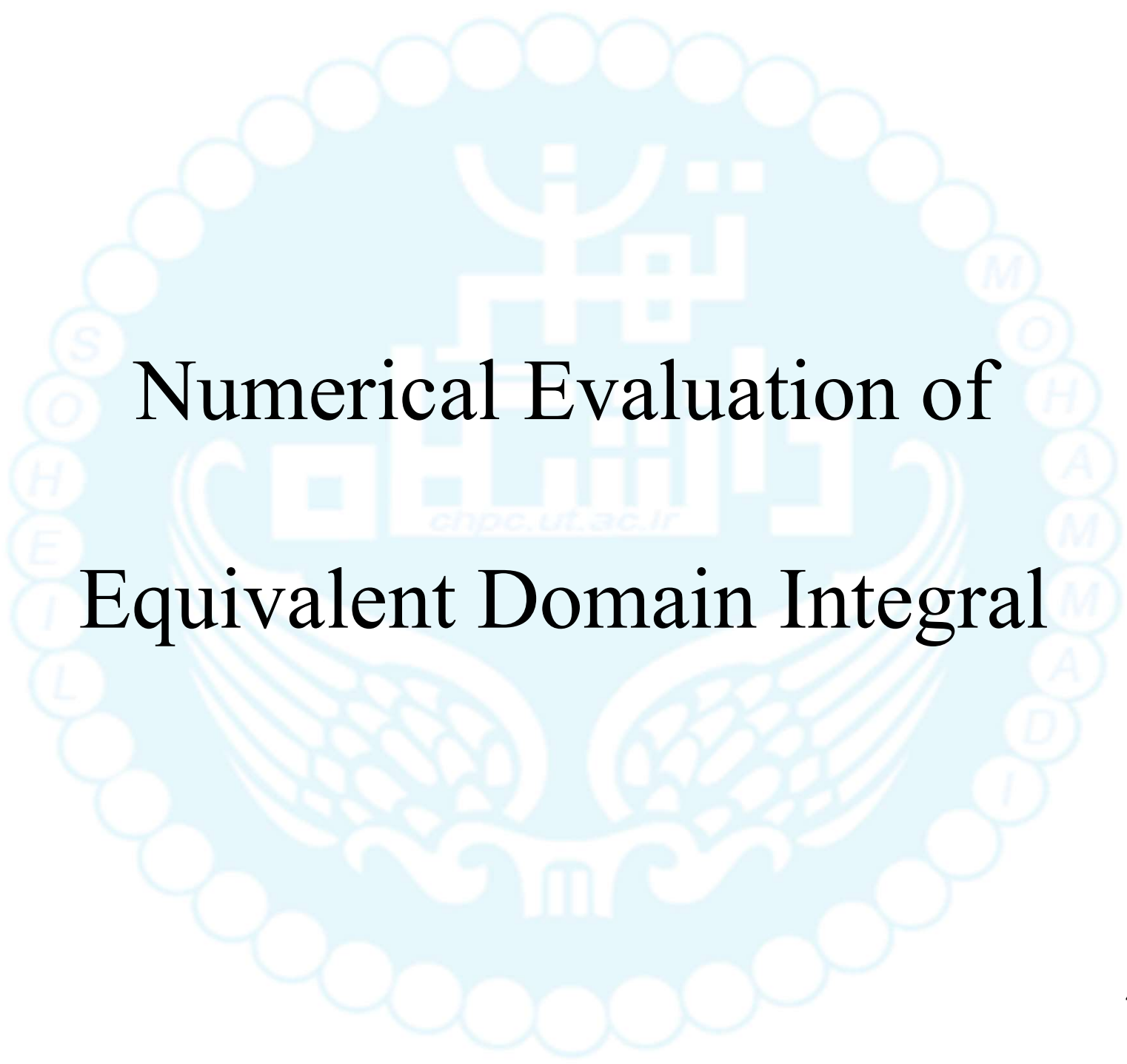
I_g is the integrand evaluated at each Gauss point g

$$J = \sum_{g=1}^{ng} W_g I_g(\xi_g, \eta_g)$$

$$I_g = \left\{ \frac{1}{2} \left[\sigma_{xx} \frac{\partial u_x}{\partial x} + \sigma_{xy} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \frac{\partial u_x}{\partial x} + \sigma_{yy} \frac{\partial u_y}{\partial y} \right] \frac{\partial y}{\partial \eta} \right. \\ \left. - \left[\left(\sigma_{xx} n_1 + \sigma_{xy} n_2 \right) \frac{\partial u_x}{\partial x} + \left(\sigma_{xy} n_1 + \sigma_{yy} n_2 \right) \frac{\partial u_y}{\partial x} \right] \sqrt{\left(\frac{\partial x}{\partial \eta} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2} \right\}_g$$

All terms of this equation are known:

- the stress components are available at the Gauss points,
- the strains can be determined from derivatives of the shape functions (**B** matrix),
- and $\partial y / \partial \eta$ is a component of the known Jacobian matrix **J**.



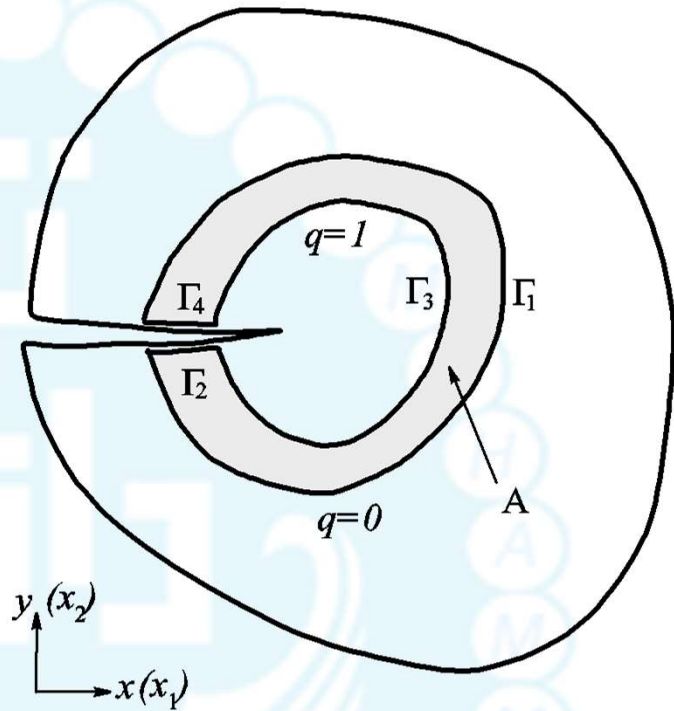
Numerical Evaluation of Equivalent Domain Integral

$$J = \int_A \left[\sigma_{ij} \frac{\partial u_i}{\partial x_j} - w_s \delta_{1j} \right] \frac{\partial q}{\partial x_j} dA$$

The value of q within an element can be interpolated as:

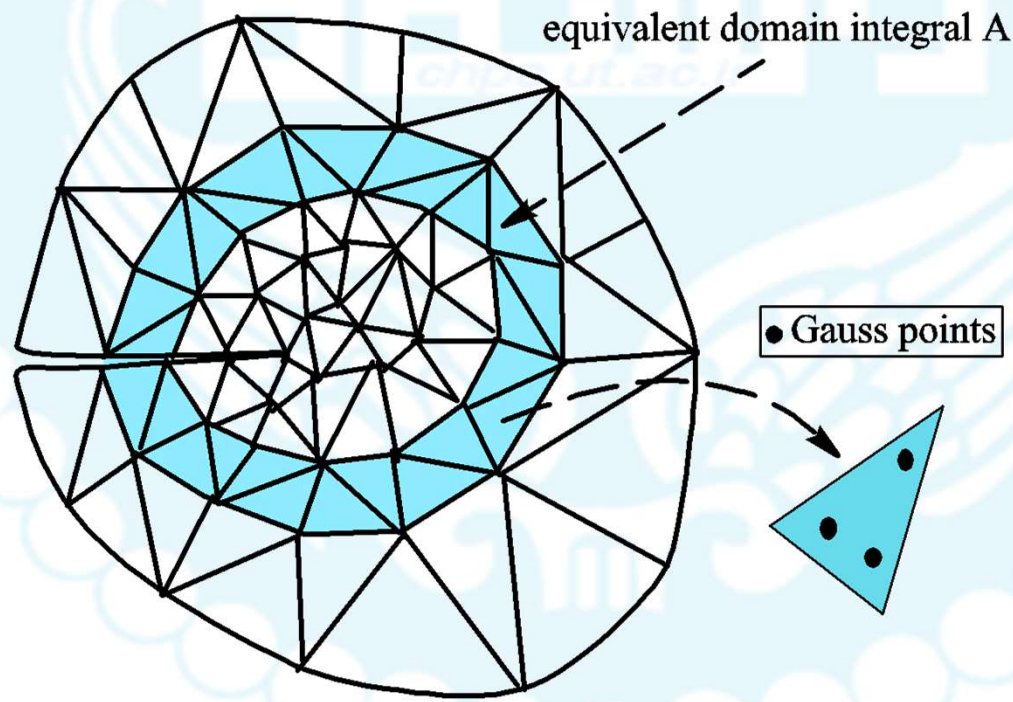
$$q = \sum_{i=1}^n N_i(x) q_i$$

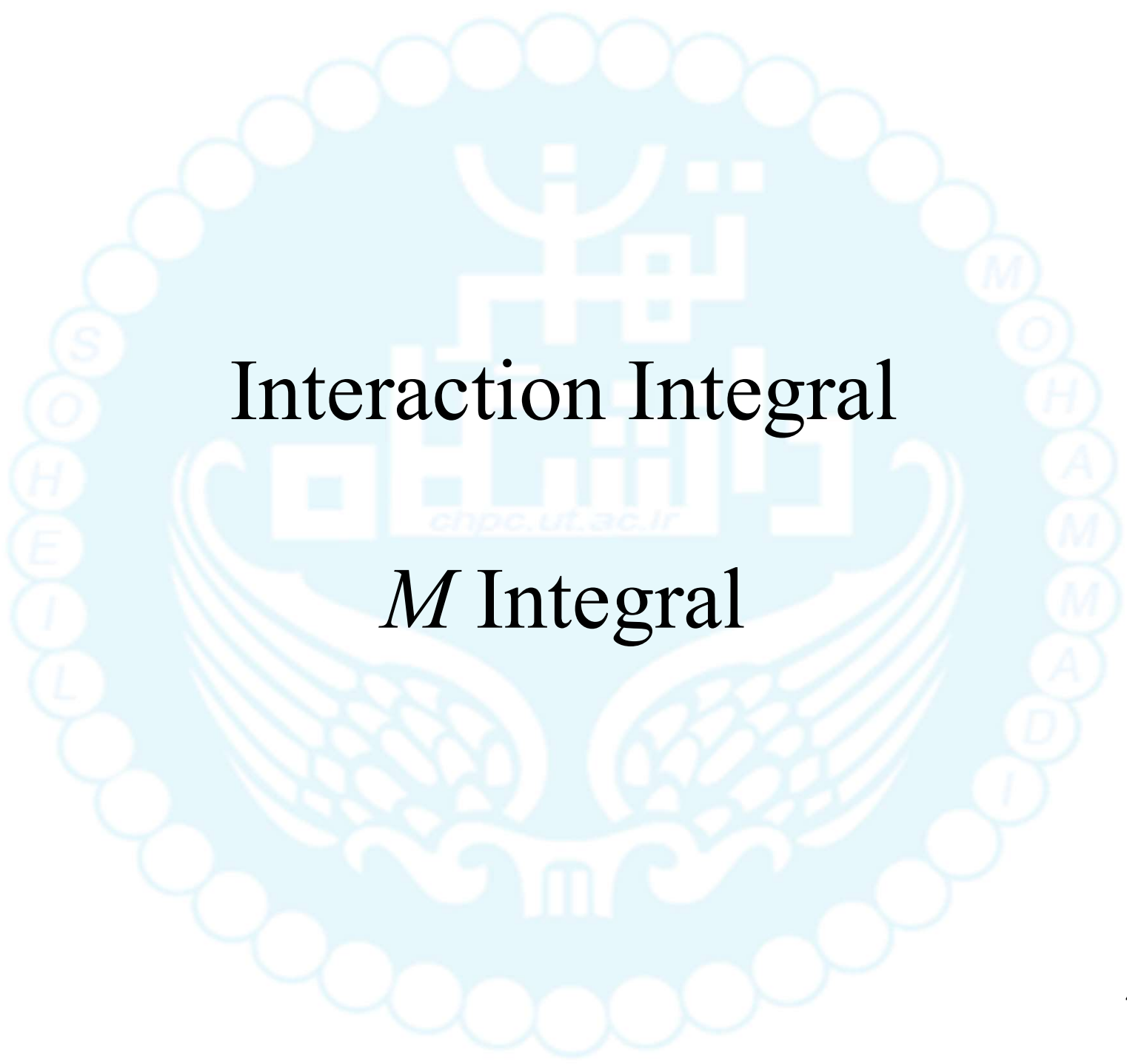
where n is the number of nodes per element, q_i are the nodal values of q , and N_i are the element shape functions.



Evaluation of J is performed by integration on the Gauss points of the elements on the equivalent domain path:

$$J = \sum_{A^*} \sum_{g=1}^{ng} \left\{ \left[\left(\sigma_{ij} \frac{\partial u_i}{\partial x_1} - w_s \delta_{1i} \right) \frac{\partial q}{\partial x_i} \right] \det \left(\frac{\partial x_j}{\partial \xi_k} \right) \right\}_g W_g$$





Interaction Integral

M Integral

Interaction integral method

The interaction integral was originally developed to compute the J integral for a combination of two independent states.

Both the original contour integral or the equivalent domain integral can be extended to derive the corresponding interaction integral formulation.

$$J = \int_{\Gamma} \left(w_s \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial x} \right) n_j d\Gamma$$
$$J = \int_A \left[\sigma_{ij} \frac{\partial u_i}{\partial x_1} - w_s \delta_{1j} \right] \frac{\partial q}{\partial x_j} dA$$

Even extensions of the J integral to include the effects of inertia, thermal, pressure, etc. can also be adopted.

Consider we have two independent states of (1) and (2) that satisfy equilibrium, and their combination (1+2). The corresponding J integrals can be defined (based on contour integral):

$$J^{(1)} = \int_{\Gamma} \left(w_s^{(1)} \delta_{1j} - \sigma_{ij}^{(1)} \frac{\partial u_i^{(1)}}{\partial x} \right) n_j d\Gamma$$

$$w_s^{(1)} = \frac{1}{2} \sigma_{ij}^{(1)} \varepsilon_{ij}^{(1)}$$

$$J^{(2)} = \int_{\Gamma} \left(w_s^{(2)} \delta_{1j} - \sigma_{ij}^{(2)} \frac{\partial u_i^{(2)}}{\partial x} \right) n_j d\Gamma$$

$$w_s^{(2)} = \frac{1}{2} \sigma_{ij}^{(2)} \varepsilon_{ij}^{(2)}$$

$$J^{(1+2)} = \int_{\Gamma} \left(w_s^{(1+2)} \delta_{1j} - \sigma_{ij}^{(1+2)} \frac{\partial u_i^{(1+2)}}{\partial x} \right) n_j d\Gamma$$

$$w_s^{(1+2)} = \frac{1}{2} \sigma_{ij}^{(1+2)} \varepsilon_{ij}^{(1+2)}$$

Using the linear elastic assumption for the displacement, strain and stress states of the combined case (1+2):

$$u_i^{(1+2)} = u_i^{(1)} + u_i^{(2)}$$

$$\varepsilon_{ij}^{(1+2)} = \varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}$$

$$\sigma_{ij}^{(1+2)} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}$$

The $J^{(1+2)}$ integral of the combined state (1+2) can then be further simplified:

$$J^{(1+2)} = \int_{\Gamma} \left[\frac{1}{2} \left(\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \right) \left(\varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)} \right) \delta_{1j} - \left(\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \right) \frac{\partial \left(u_i^{(1)} + u_i^{(2)} \right)}{\partial x} \right] n_j d\Gamma$$

Separating the terms of $J^{(1)}$ and $J^{(2)}$, the remaining terms are called the M integral

$$J^{(1+2)} = J^{(1)} + J^{(2)} + \underbrace{M}_{\text{remaining terms}}$$

$$M = \int_{\Gamma} \left[\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - w^M \delta_{1j} \right] n_j d\Gamma$$

$$w^M = \frac{1}{2} \left(\sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} + \sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)} \right)$$

Interaction integral method for SIFs computation

In special circumstances, the concept of interaction integral for two combined states can be used to determine the stress intensity factors of any mixed mode problem.

This is achieved by introducing the auxiliary fields and superimposing them onto the actual fields that satisfying the boundary value problem (Sih *et al.*, 1965).

Stresses and strains for the auxiliary state should be chosen so as to satisfy both the equilibrium equation and the traction free boundary condition on the crack surface in the A area.

The auxiliary fields are suitably selected in order to find a relationship between the mixed mode stress intensity factors and the interaction integrals.

The J integral for the sum of the two states can be defined as

$$J = J^{\text{act}} + J^{\text{aux}} + M$$

where J^{act} and J^{aux} are associated with the actual and auxiliary states, respectively, and M is the interaction integral (based on EDI formulation):

$$J^{\text{act}} = \int_A \left[\sigma_{ij} \frac{\partial u_i}{\partial x_1} - w_s \delta_{1j} \right] \frac{\partial q}{\partial x_j} dA$$

$$J^{\text{aux}} = \int_A \left[\sigma_{ij}^{\text{aux}} \frac{\partial u_i^{\text{aux}}}{\partial x_1} - w^{\text{aux}} \delta_{1j} \right] \frac{\partial q}{\partial x_j} dA$$

$$M = \int_A \left[\sigma_{ij} \frac{\partial u_i^{\text{aux}}}{\partial x_1} + \sigma_{ij}^{\text{aux}} \frac{\partial u_i}{\partial x_1} - w^{\text{M}} \delta_{1j} \right] \frac{\partial q}{\partial x_j} dA$$

with the actual, auxiliary and interaction works defined as:

$$w_s = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$$

$$w^{\text{aux}} = \frac{1}{2} \sigma_{ij}^{\text{aux}} \varepsilon_{ij}^{\text{aux}}$$

$$w^{\text{M}} = \frac{1}{2} \left(\sigma_{ij} \varepsilon_{ij}^{\text{aux}} + \sigma_{ij}^{\text{aux}} \varepsilon_{ij} \right)$$

One of the choices for the auxiliary state is the displacement and stress fields in the vicinity of the crack tip (such as the Westergaard solution). From the relation of the J integral and mode I and II stress intensity factors,

$$J = \frac{1}{E'} (K_{\text{I}}^2 + K_{\text{II}}^2)$$

the following relationship is obtained

$$M = \frac{2}{E'} (K_I K_I^{\text{aux}} + K_{II} K_{II}^{\text{aux}})$$

Therefore, the mode I and II stress intensity factors can be obtained from two post processing computations (with specific assumptions):

$$\text{case1 : } K_I^{\text{aux}} = 1, K_{II}^{\text{aux}} = 0 \rightarrow M^{\text{case1}} \rightarrow K_I = \frac{E'}{2} M^{\text{case1}}$$

$$\text{case2 : } K_I^{\text{aux}} = 0, K_{II}^{\text{aux}} = 1 \rightarrow M^{\text{case2}} \rightarrow K_{II} = \frac{E'}{2} M^{\text{case2}}$$

It should be noted that evaluation of the interaction integral requires careful attention as the main fields are usually obtained from the finite element solution in a global or local (but crack-independent) coordinate system, while the auxiliary fields are defined in the local crack tip polar coordinate system. Therefore, necessary transformations are required to use a unified coordinate system.

Computing SIFs by this approach involves two fast postprocessing M integral calculations based on FEM known solutions of displacements, strains and stresses. It does not require any additional expensive solution of equilibrium equations.

This method is the most advanced and accurate technique for computing SIFs in virtually any fracture mechanics problem.



J-Integral and Plasticity

***J* integral for EPFM**

Although the *J* integral was originally developed for linear elastic problems, it can be used for nonlinear monotonic problems, where no unloading or reverse loading is allowed.

For details see Anderson (1994).

Plastic crack tip fields

Whereas under LEFM assumptions, we had a $r^{-0.5}$ stress and strain singularity, those singularities may be determined for a simple type of EPFM.

In order to illustrate how the concept of the J integral can be used in nonlinear analysis, a typical elastoplastic problem based on the Ramberg–Osgood power hardening law is considered (Hutchinson 1968, 1990),

$$\frac{\varepsilon}{\varepsilon_{yld}} = \frac{\sigma}{\sigma_{yld}} + k_0 \left(\frac{\sigma}{\sigma_{yld}} \right)^n$$

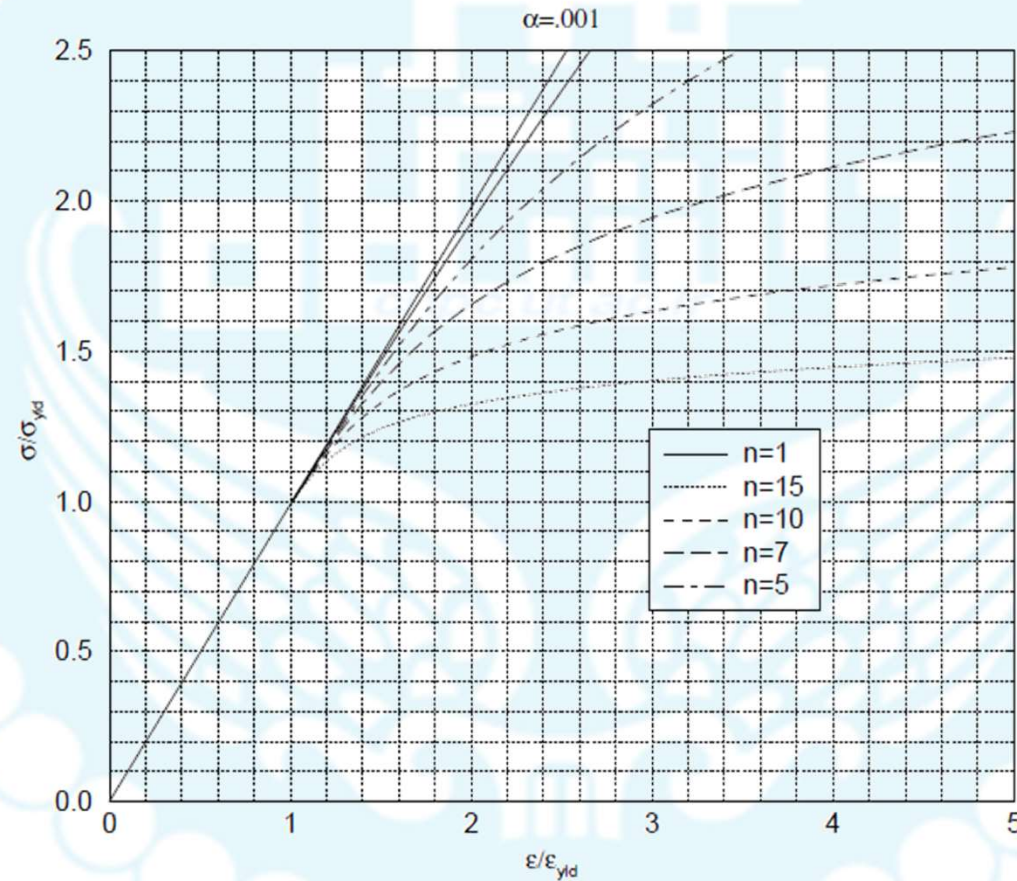
n is the strain hardening exponent and k_0 is a dimensionless constant.

$$\frac{\varepsilon}{\varepsilon_{yld}} = \frac{\sigma}{\sigma_{yld}} + k_0 \left(\frac{\sigma}{\sigma_{yld}} \right)^n$$

$n = 1$: a linear elastic response

$n = \infty$: an elastic perfectly plastic

Ramberg-Osgood Stress-Strain Relation



Normalize Ramberg-Osgood Stress-Strain Relation ($k_0 = .01$)

Neglecting the elastic strain in the vicinity of the crack tip (because the plastic strain is dominant near the crack tip), leads to:

$$\frac{\varepsilon}{\varepsilon_{yld}} = k_0 \left(\frac{\sigma}{\sigma_{yld}} \right)^n$$

$$\varepsilon(r) = \frac{k_1}{r^{\lambda_\varepsilon}}, \quad \lambda_\varepsilon = \frac{n}{n+1}$$

$$\sigma(r) = \frac{k_2}{r^{\lambda_\sigma}}, \quad \lambda_\sigma = \frac{1}{n+1}$$

or

$$\varepsilon(r) = k_3 \left(\frac{J}{r} \right)^{\frac{1}{n+1}}$$

$$\sigma(r) = k_4 \left(\frac{J}{r} \right)^{\frac{n}{n+1}}$$

Note different orders of singularity of stress and strain states.⁶⁰

The relationship between J and the crack tip displacement, stress and strain fields can be expressed as:

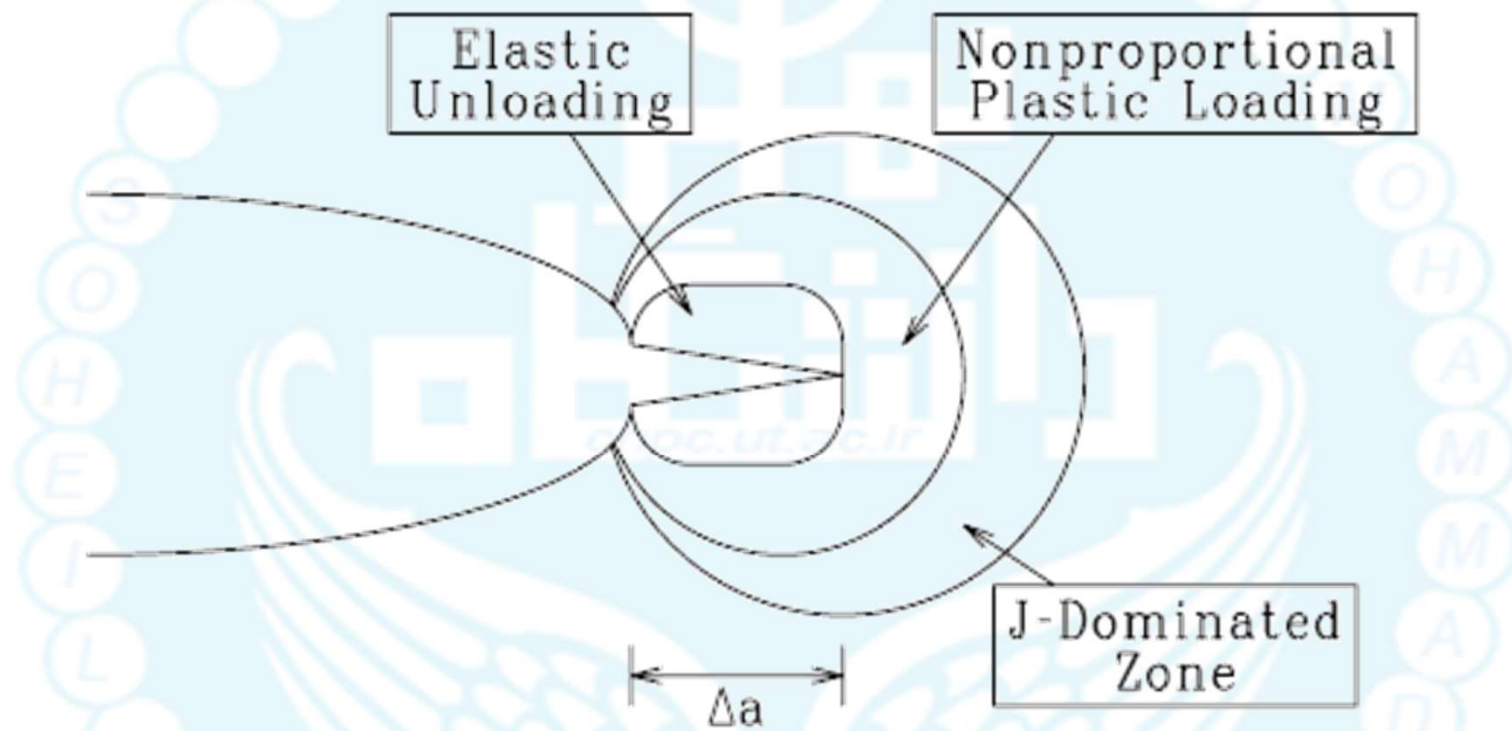
$$u_i(r, \theta) = k_0 \varepsilon_{yld} r \left(\frac{J}{k_0 \sigma_{yld} \varepsilon_{yld} r I_n} \right)^{\frac{n}{n+1}} \bar{u}_i(\theta, n)$$

$$\sigma_{ij}(r, \theta) = \sigma_{yld} \left(\frac{EJ}{k_0 \sigma_{yld}^2 r I_n} \right)^{\frac{1}{n+1}} \bar{\sigma}_{ij}(\theta, n)$$

$$\varepsilon_{ij}(r, \theta) = \frac{k_0 \sigma_{yld}}{E} \left(\frac{EJ}{k_0 \sigma_{yld}^2 r I_n} \right)^{\frac{n}{n+1}} \bar{\varepsilon}_{ij}(\theta, n)$$

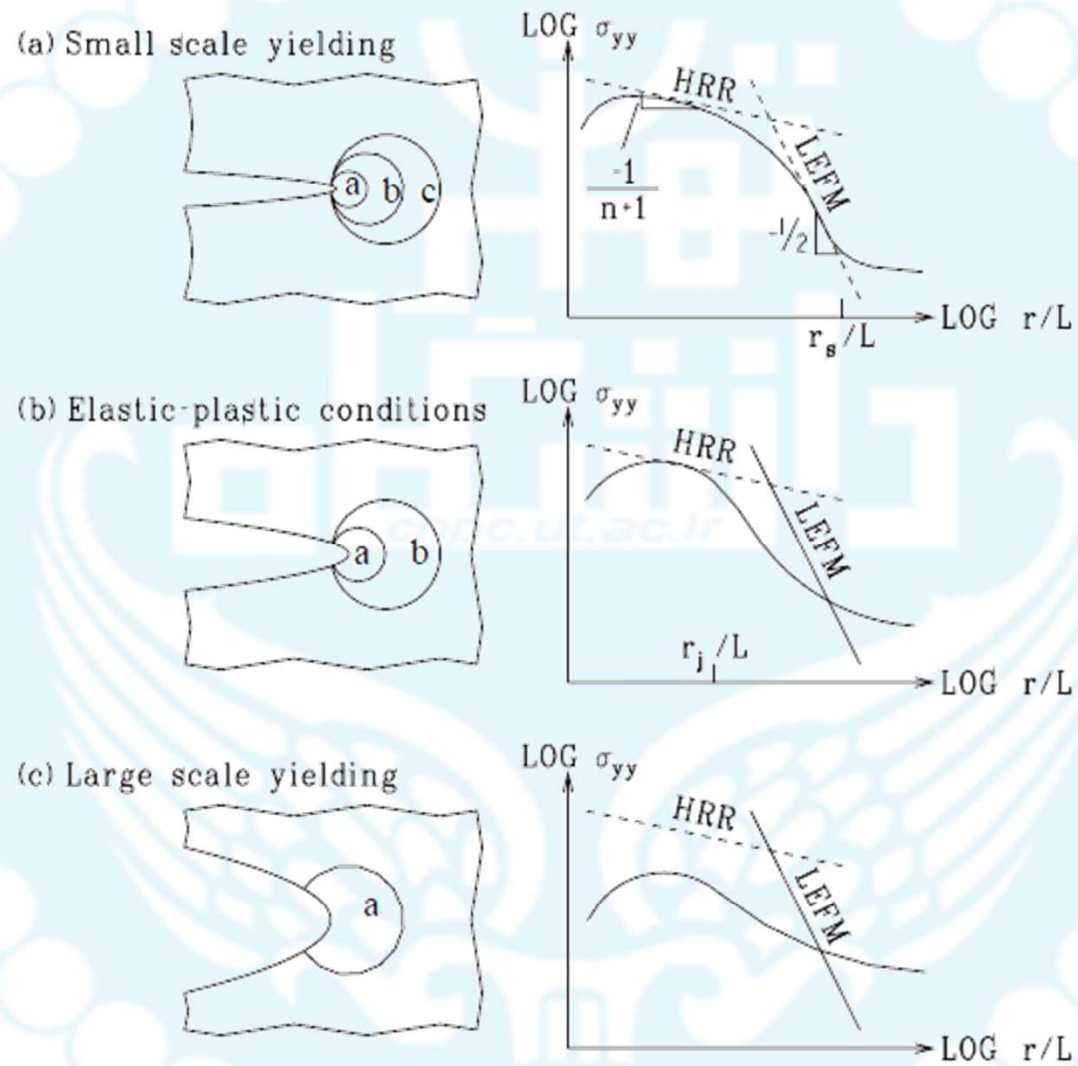
I_n is an integration constant which depends on the stress–strain curve.

σ and ε are dimensionless geometric functions of n and θ .



HRR Singularity, (Anderson 1995)

a Large Strain Region b: J-dominated zone; c: K-dominated zone



Effect of Plasticity on the Crack Tip Stress Fields, (Anderson 1995)

Effect of Plasticity on the Crack Tip Stress Field, (Anderson 1995)

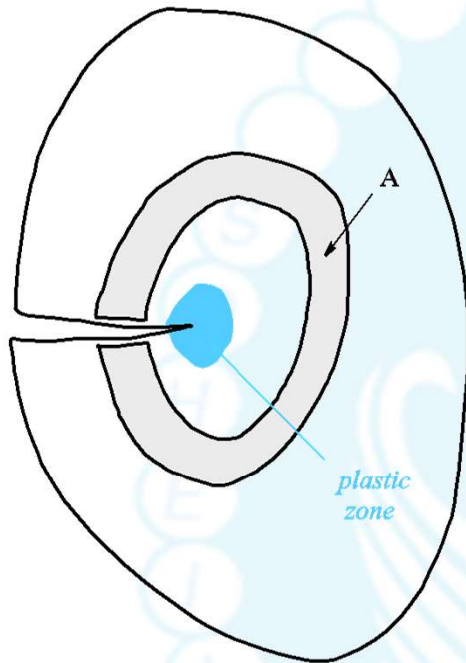
	Large Strain	J Dominated	K Dominated
Small scale yielding	Y	Y	Y
Elastic Plastic Conditions	Y	Y	N
Large Scale Yielding	Y	N	N

Small Scale Yielding: both K and J characterize this zone.

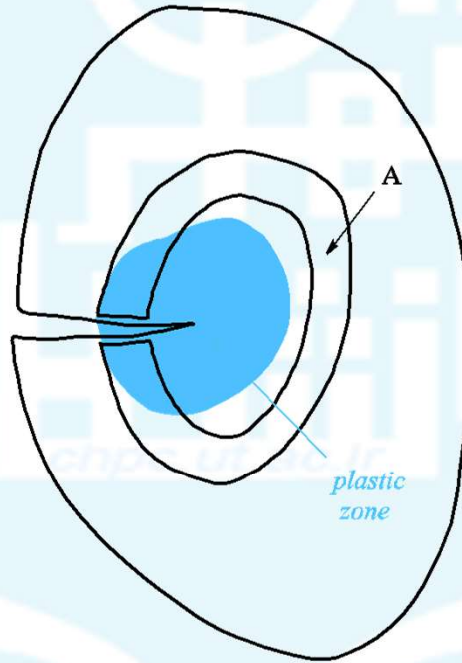
1. At a short distance from the crack tip, the stress is proportional to $r^{-0.5}$ and this area is called the K -dominated region.
2. If we have monotonic loading, a J -dominated region occurs in the plastic zone where the elastic singularity is no longer valid. Inside the plastic zone, the HRR is approximately valid.
3. Finally, finite strain region occurs within approximately 2δ from the crack tip where the large deformation invalidates HRR.

Elastic-Plastic: J is still valid, but K no longer.

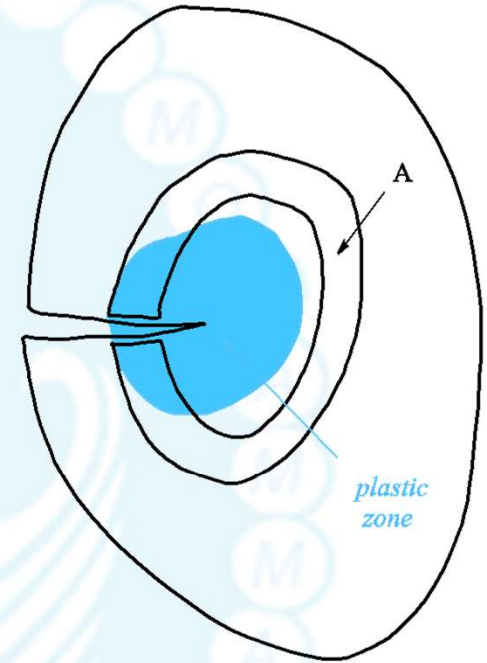
Large Scale Yielding: the size of the finite strain zone becomes significant and there is no longer a region uniquely characterized by J as it becomes size and geometry dependent.



K for small plastic zone



J for monotonic loading



G for general loading/unloading

